Brownian rectifiers in the presence of temporally asymmetric unbiased forces

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The efficiency of energy transduction in a temporally asymmetric rocked ratchet is studied. Time asymmetry favors current in one direction and suppresses it in the opposite direction due to which large efficiency $\sim 50\%$ is readily obtained. The spatial asymmetry in the potential together with system inhomogeneity may help in further enhancing the efficiency. Fine tuning of system parameters considered leads to multiple current reversals even in the adiabatic regime.

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I. INTRODUCTION

Brownian rectifiers or ratchets are devices that convert nonequilibrium fluctuations into useful work in the presence of load. Several physical models [1–4] have been proposed to understand the nature of currents and their possible reversals with applications in nanoparticle separation devices [4]. The possibility of enhancement of efficiency with which these Brownian rectifiers convert the nonequilibrium fluctuations into useful work has generated much interest in this field. This, in turn, has led to the emergence of a separate subfield—stochastic energetics—on its own right [5,6]. Using this formalism one can readily establish the compatibility between Langevin or Fokker–Planck formalism with the laws of thermodymanics thereby providing a tool to study systems far from equilibrium. With this framework, one can calculate various physical quantities such as efficiency of energy transduction [7], energy dissipation (hysteresis loss), entropy (entropy production) [8], etc.

Most of the studies yield low efficiencies, in the subpercentage range, in various types of ratchets. This is due to the intrinsic irreversibility associated with ratchet operation. Only fine tuning of parameters can lead to a large efficiency, the regime of parameters, however, being very narrow [9]. Recently Makhnovskii *et al.* [10] constructed a special type of flashing ratchet with two asymmetric double-well periodic-potential states displaced by half a period. Such flashing ratchet models were found to be highly efficient with efficiency an order of magnitude higher than in earlier models [5–7,11]. The basic idea behind this enhanced efficiency is that even for diffusive Brownian motion the choice of appropriate potential profile ensures suppression of backward motion and hence reduction in the accompanying dissipation.

In the present work, we study the motion of a particle in a rocking ratchet rocked purposefully as to favor current in one direction but to suppress motion in the opposite direction. This is in similar spirit as in case of flashing ratchets proposed by Makhnovskii *et al.* [10]. This is accomplished by applying temporally asymmetric but unbiased periodic forcings [12–14]. Interestingly, such choice of forcings help in obtaining rectified currents with high efficiency even for spatially symmetric periodic potentials. Still higher efficiency is obtained with asymmetric potentials. The range of parameters of operation of such ratchets is quite wide sustaining large loads. In addition, frictional inhomogeneity may further enhance the efficiency. Our study is closely related to Ref. [14]. However, there is an error in the calculation of input energy in Ref. [14] which we have rectified [15]. Due to this our results cannot be compared with that in Ref. [14]. We also see multiple current reversals in the full parameter space of operation even in the adiabatic regime. However, multiple current reversals require fine tuning of the parameters.

II. MODEL

The Brownian motion of a particle in an inhomogeneous medium is described by the Langevin equation [16]

$$\dot{q} = -\frac{V'(q) - F(t)}{\gamma(q)} - \frac{k_B T \gamma'(q)}{2[\gamma(q)]^2} + \sqrt{\frac{k_B T}{\gamma(q)}}\xi(t), \qquad (1)$$

where $\xi(t)$ is a randomly fluctuating Gaussian thermal noise with zero mean and correlation, $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')$. It may be noted that Eq. (1) has been obtained earlier via a microscopic treatment of a system coupled to a thermal bath [16]. The above equation involves a multiplicative noise with an additional temperature dependent drift term. This additional drift term is essential in order for the system to approach the correct thermal equilibrium state in the absence of forcing [16,17]. The periodic potential $V(q) = -V_0 \sin(q) - (\mu/4)\sin(2q)$. The parameter $\mu(-1<\mu<1)$ characterizes the degree of asymmetry in the potential. The friction coefficient $\gamma(q) = \gamma_0(1-\lambda \sin(q+\phi))$, with $0 \leqslant \lambda < 1$ and ϕ the phase difference. $F(t)$ is the externally applied periodic driving force. Following Stratonovich interpretation [18] the corresponding Fokker-Planck equation [19] is given by

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$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\frac{1}{\gamma(q)}\left[k_B T\frac{\partial P(q,t)}{\partial q} + [V'(q) - F(t)]P(q,t)\right]. \tag{2}$$

Since we are interested in the adiabatic limit, we first obtain an expression for the probability current density j in the presence of a constant external force F_0. The expression is given by [19]

$$j = \frac{1 - \exp\left[\frac{-2\pi F_0}{k_B T}\right]}{\int_0^{2\pi} dy I_-(y)}, \tag{3}$$

where $I_-(y)$ is given by

$$I_-(y) = \exp\left[\frac{-V(y) + F_0 y}{k_B T}\right]$$

$$\int_y^{y+2\pi} dx\, \gamma(x)\exp\left[\frac{V(x) - F_0 x}{k_B T}\right]. \tag{4}$$

It may be noted that for $\mu=0$, $j(F_0)\neq -j(-F_0)$ for $\phi\neq 0,\pi$. This asymmetry ensures rectification of current for the rocked ratchet even in the presence of spatially symmetric potential. We assume that $F(t)$ changes slow enough, i.e., its frequency is smaller than any other frequency related to the relaxation rate in the problem such that the system is in a steady state at each instant of time.

We consider time asymmetric ratchets with a zero-mean periodic driving force [12] given by

$$F(t) = \frac{1+\epsilon}{1-\epsilon}F_0, \quad \left(n\tau \leqslant t < n\tau + \frac{1}{2}\tau(1-\epsilon)\right),$$
$$= -F_0, \quad \left(n\tau + \frac{1}{2}\tau(1-\epsilon) < t \leqslant (n+1)\tau\right). \tag{5}$$

Here, the parameter ϵ signifies the temporal asymmetry in the periodic forcing, τ is the period of the driving force $F(t)$ and $n=0,1,2\ldots.$ is an integer. For this forcing in the adiabatic limit, the expression for time averaged current is [7,12]

$$\langle j\rangle = j^+ + j^-, \tag{6}$$

with

$$j^+ = \frac{1}{2}(1-\epsilon)j\left(\frac{1+\epsilon}{1-\epsilon}F_0\right),$$

$$j^- = \frac{1}{2}(1+\epsilon)j(-F_0) \tag{7}$$

where j^+ is the current fraction in the positive direction over a fraction of time period $(1-\epsilon)/2$ of τ when the external driving force field is $(1+\epsilon/1-\epsilon)F_0$ and j^- is the current fraction over the time period $(1+\epsilon)/2$ of τ when the external driving force field is $-F_0$. The input energy E_{in} per unit time is given by [7]

$$E_{\text{in}} = F_0\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)j^+ - j^-\right]. \tag{8}$$

To calculate efficiency a load L is applied against the direction of current with overall potential $V(q)=-[V_0\sin(q) + (\mu/4)\sin(2q) - qL]$. The current flows against the load as long as the load is less than the stopping force L_s beyond which the current is in the same direction as that of the load. Thus, in the operating range of the load $0<L<L_s$, the Brownian particles move in the direction opposite to the load thereby storing energy. The average work done over a period is given by

$$E_{\text{out}} = L[j^+ + j^-]. \tag{9}$$

The thermodynamic efficiency of energy transduction is [5,6]

$$\eta = \frac{L[j^+ + j^-]}{F_0\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)j^+ - j^-\right]}. \tag{10}$$

In our present discussion, all the physical quantities are taken in dimensionless units. In the following section, we discuss the results of our calculation. In order to evaluate currents, we use the method of Gaussian quadrature [20].

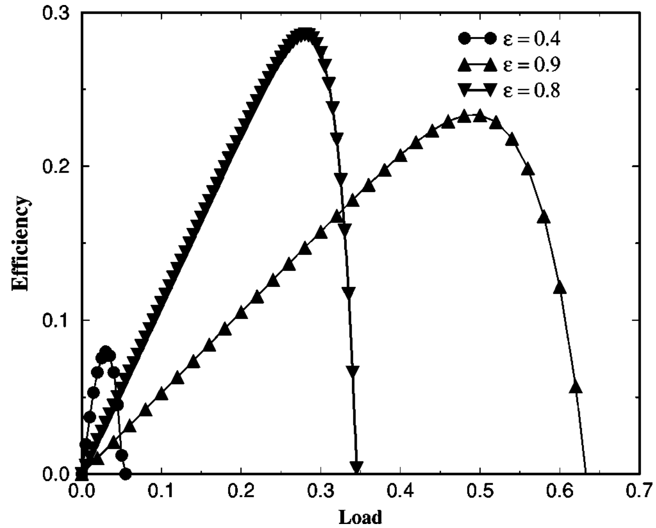

FIG. 1. Efficiency vs load for various values of ϵ with fixed $F_0=0.1$, $\mu=0$, and $\lambda=0$.

III. RESULTS AND DISCUSSIONS

To begin with, we consider a homogeneous system in the presence of spatially symmetric potential. In this case, unidirectional currents arise solely due to temporally asymmetric driving field (with mean zero) which is mainly emphasized in this work.

In Fig. 1, we plot efficiency as a function of load in the presence of temporal asymmetry ϵ for $F_0=0.1$ and $T=0.1$. For finite ϵ, current against the load is obtained for load $L<L_s$. The stopping force L_s is an increasing function of ϵ. As will be discussed later (in Fig. 2), the total current increases as ϵ increases and hence larger load is necessary to

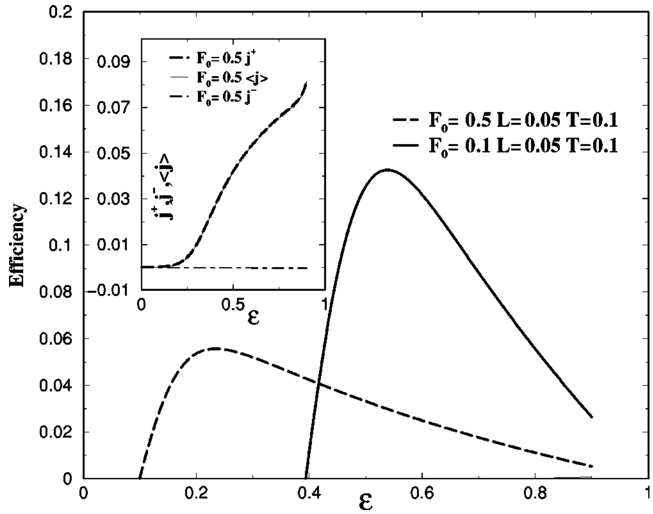

FIG. 2. Efficiency vs ϵ for $F_0=0.1$ and 0.5 for $T=0.1$ with fixed $\mu=0$ and $\lambda=0$. Inset shows j^+, j^- and $\langle j\rangle$ for $F_0=0.5$ with other parameters being the same.

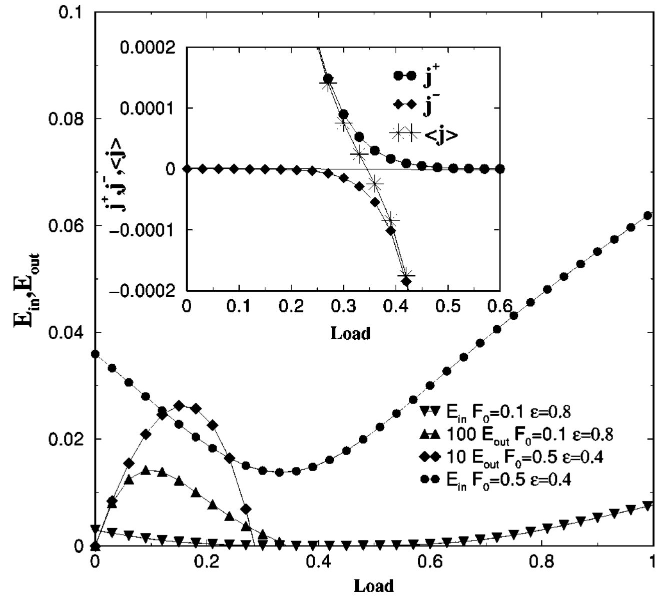

FIG. 3. Input and output energy vs load for $\epsilon=0.8$ and $F_0=0.1$, and $\epsilon=0.4$ and $F_0=0.5$ with fixed $\mu=0$, $\lambda=0$, and $T=0.1$. The negative values of the output energy are not shown. The output curve is blown up 100 and 10 times, respectively, to scale with the input energy curve values. Inset shows the currents, j^+, j^- and $\langle j\rangle$ for $\epsilon=0.8$ and $F_0=0.1$ as a function of load.

reverse the direction of current. For a given ϵ, the efficiency slowly increases with load, attains a maximum and then decreases rapidly (Fig. 1). The locus of peak values corresponding to different ϵ values (with appropriate load L) is found to have a nonmonotonous behavior with the maximum (~ 0.29) being at around $\epsilon=0.8$ corresponding to a load of $L=0.258$. It may be emphasized that an efficiency above the subpercentage value is readily obtained here as compared to the reported values in other ratchet models [7]. Moreover, this ratchet system can sustain larger operation range of load against which useful work is done by pumping particles uphill.

The existence of the optimal value of ϵ can be understood as follows. At low temperatures, with k_BT much less than V_0, the modulation amplitude of the periodic symmetric potential (which is taken to be unity in our present problem as all the energies are scaled with respect to V_0), the significant current arises only when the bias field is greater than the critical field F_c, which should be greater than one for our case [19]. It may be noted that j^+ and j^- are always positive and negative, respectively. If $F_0<1$, the current fraction in the negative direction is very small (blocking of current). The significant current fraction in the positive direction arise only when $(1+\epsilon/1-\epsilon)F_0>1$. In this situation, the barriers for motion in the forward direction disappears. For fixed F_0, the above condition determines the critical value of ϵ, i.e., ϵ_c. Thus, when $\epsilon>\epsilon_c$, the barrier to forward motion disappears and hence $j^+ \gg j^-$ leading to

$$\eta=\frac{L(1-\epsilon)}{F_0(1+\epsilon)}. \tag{11}$$

It may be noted that as $\epsilon\to 1$, $\eta\to 0$ from the positive side. To generate useful work in the adiabatic domain the load has to be smaller than the largest applied field $(1+\epsilon/1-\epsilon)F_0$ in the positive direction which guarantees that $\eta<1$. In the opposite limit of $\epsilon\to 0$, $j^+\sim -j^-$ (in the presence of infinitesimal load) and hence no useful work can be obtained (E_{out} ~ 0). These two limiting cases explain the existence of an optimum value for ϵ. In the limit $k_BT\ll V_0$ and $F_0\ll 1$, this optimal value ϵ_c can be found from the approximate condition $(1+\epsilon_c/1-\epsilon_c)F_0=1$. For the present case of Fig. 1 with $F_0=0.1$, the above expression gives $\epsilon_c=0.82$ and corresponding efficiency as ~ 0.26 in reasonable agreement with that in figure. The critical value of ϵ_c depends on F_0 and it decreases with increase in F_0.

To make the above discussion transparent in Fig. 2, we plot efficiency as a function of ϵ for two values of F_0; namely, $F_0=0.1$ and 0.5. The larger F_0 value, i.e., $F_0=0.5$ is taken so as to observe efficiency even at lower ϵ values. It is clear that efficiency shows a nonmonotonous behavior with ϵ as discussed above and ϵ_c follows the expected behavior with respect to increase in F_0. The inset shows j^+ and j^- as a function of ϵ for the case $F_0=0.5$. The suppression of backward current fraction is obvious in this figure. It is this suppression that leads to larger efficiency consistent with the observations made in [10]. The forward fraction of current increases monotonically with ϵ. Plots of j^+ and $\langle j\rangle$ are merged together on the scale used in the plot.

The useful work E_{out} and the input energy E_{in} are shown in Fig. 3 for two representative values of ϵ, namely, $\epsilon=0.8$ and $\epsilon=0.4$ as a function of load. The input energy decreases to a minimum value for a load larger than L_s. This minimum corresponds to a value of about 2.497×10^{-5} for a load of 0.399 when $\epsilon=0.8$. Moreover, it remains positive, as expected, over the entire range. The output energy shows a peak with load in the region where the input energy is monotonously decreasing. It then becomes negative for $L>L_s$ as anticipated. This qualitative behavior of input energy shown in Fig. 3 is similar to that in Ref. [10] for the flashing

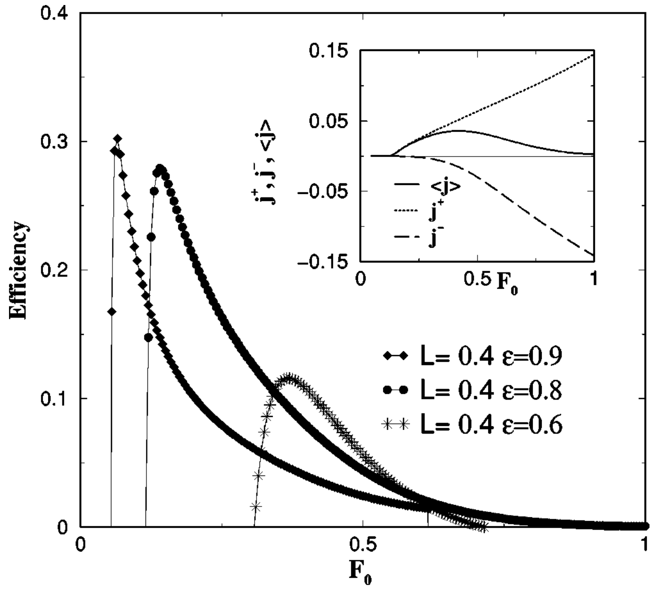

FIG. 4. Efficiency vs F_0 for $\epsilon=0.9$, 0.8, and 0.6 with fixed $\mu=0$, $\lambda=0$, $L=0.4$, and $T=0.1$.

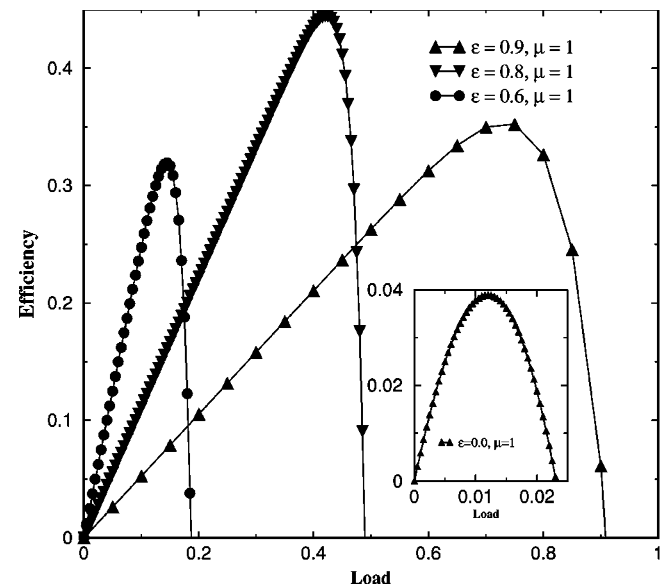

FIG. 5. Efficiency vs load for various ϵ with fixed $F_0=0.1$, $T=0.1$, and $\lambda=0$. Inset shows efficiency vs load for the case $\epsilon=0$ and $\mu=1$.

ratchet. Needless to say, that finite input energy is always required to maintain the nonequilibrium steady state.

The behavior of input energy can be explained from the nature of current fractions j^+ and j^- which is shown in the inset of Fig. 3. Up to the point where the load $L\sim 0.26$, the current in the backward direction, i.e., j^- is negligible and hence the contribution to input energy is solely due to the current fraction in the positive direction. Hence, there is a monotonic decrease of E_{in} in accordance with j^+ with increasing load as is obvious from the equation for input energy, Eq. (8), which is now $E_{\text{in}}\sim F_0\big((1+\epsilon)/(1-\epsilon)\big)j^+$. Up to this load, the net unidirectional current $\langle j\rangle$ coincides with j^+. With a still further increase in load, the current fraction in the backward direction (with negative magnitude) starts increasing and at a load of $L\sim 0.343$ the net unidirectional current, $\langle j\rangle$, reverses sign. Beyond this region, the major contribution to $\langle j\rangle$ comes from j^-. Since j^- is negative it follows from Eq. (8) that input energy remains positive and increases further. Again beyond a certain value of load, j^+ also reverses its sign (not shown in Fig. 3). However, as $|j^-|>|j^+|$, it follows that input energy is positive. Thus, a minima in the input energy is naturally expected with increasing load due to the balance between the current fraction in the positive and negative direction.

In Fig. 4, we plot efficiency as a function of F_0 for different values of ϵ. All other parameters are mentioned in figure caption. In the regime $k_BT\ll V_0$, the efficiency shows a peaking behavior as a function of F_0. In the limit $F_0\to 0$, there exist barriers to motion in both directions due to which transitions out of the well in either direction are suppressed and hence both net current and efficiency tends to zero. Again, when F_0 becomes large such that barriers to motion in either direction disappear, the net current again vanishes and the efficiency tends to zero. From the nature of j^+, j^-, and $\langle j\rangle$ shown in the inset the above mentioned points become more clear. The peak in the efficiency shifts to lower F_0 values with increase in ϵ as anticipated. The observed efficiency values are also quite large.

So far, our discussion was restricted to the case of spatially symmetric potential. It is the temporally asymmetric fluctuations (with zero mean) alone that lead to unidirectional current with large efficiency. In the following, we discuss briefly how the addition of asymmetry in the potential and system inhomogeniety affect the system efficiency and currents. In Fig. 5, we consider the periodic potential $V(q)$ to be spatially asymmetric together with a temporally asymmetric external driving force field. The potential asymmetry enhances the efficiency of energy transduction as well as widens the range of load. This is due to the fact that for $\epsilon\neq 0$, the presence of asymmetric parameter $\mu(>0)$ further reduces the potential barrier for forward motion and enhances the barrier for backward motion. Moreover, as can be seen from the inset, one can get finite current even when $\epsilon=0$ with finite stopping force L_s in contrast to the symmetric potential case. From Figs. 1 and 5, it is clear that the temporally asymmetric forces not only enhance the efficiency of energy transduction but also widen the operation range of load against which the ratchet system works. Specifically, we would like to emphasize that the spatially asymmetric potential enhances the energy transduction because we have chosen $\mu>0$, i.e., we apply the largest force to the soft direction of the potential. Obviously, the opposite results hold when $\mu<0$.

In Fig. 6, we plot efficiency as a function of T for various ϵ values in the presence of potential asymmetry $(\mu>0)$. The efficiency decreases with temperature. The efficiency approaches the value $[L(1-\epsilon)]/[F_0(1+\epsilon)]$ in the low-temperature limit, Eq. (11). This gives an efficiency value of 0.275, 0.385, and 0.488 for cases (i), (ii), and (iii), respectively, of Fig. 6. These values are consistent with that obtained in Fig. 6. The relevant physical parameters chosen for optimal efficiency are mentioned in the caption. From the inset, it should be noted that the current peaks as a function

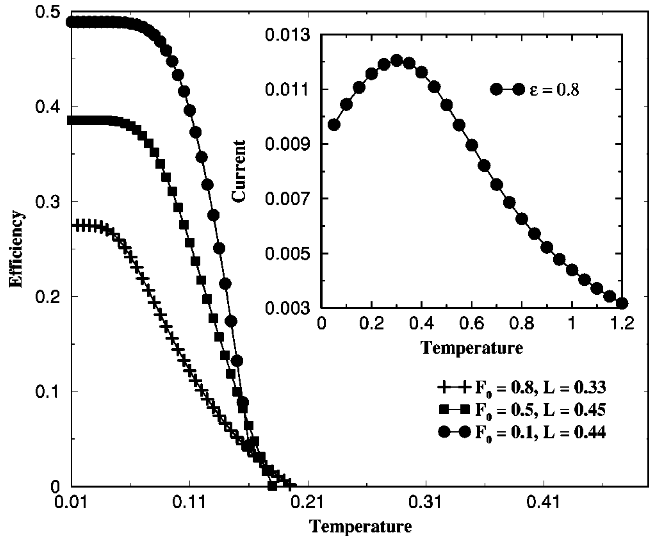

FIG. 6. Efficiency vs temperature with fixed $\mu=1$ and $\lambda=0$ for (i) $\epsilon=0.2$, $F_0=0.8$, and $L=0.33$, (ii) $\epsilon=0.4$, $F_0=0.5$, and $L=0.45$ and (iii) $\epsilon=0.8$, $F_0=0.1$, and $L=0.44$. Inset shows current as a function of temperature for $\epsilon=0.8$ and $F_0=0.1$ in the absence of load.

of temperature yet efficiency decreases monotonically. This implies that thermal fluctuation does not favor energy transduction in the case where $\mu>0$. We have verified separately that $|j^-/j^+|$ is a monotonically increasing function of temperature. This fact alone [7] along with Eq. (10) will lead to a conclusion [7] that efficiency decreases with rise in temperature. In the presence of system inhomogeniety by fine tuning the parameters, we have observed a peak in efficiency as a function of temperature as is observed in [7,21]. The observed efficiency values are in the subpercentage regime. Moreover, peaking of efficiency as a function of temperature can be readily observed when the system exhibits multiple current reversals [21]. These results are not presented here.

Next, we present the effect of frictional inhomogeneity $(\gamma=\gamma(q);\lambda\neq 0)$. In Fig. 7, we plot the efficiency as a function of the phase difference between the potential and the friction coefficient $\gamma(q)$ for a typical case. We observe that the inclusion of this parameter λ further increases the efficiency in a range of ϕ depending on other parameter values. It is worth mentioning that for inhomogeneous systems the efficiency peaks with temperature in a limited range of parameters. With frictional inhomogeneity, the range of temperatures in which one can obtain output current with finite efficiency is extended to a large temperature where the contribution of λ dominates over other parameters.

In Fig. 8, we show that by properly choosing the parameters we can obtain multiple current reversals as a function of temperature. It should be noted that such reversals are not possible in the homogeneous case in the adiabatic regime [22]. The inset of Fig. 8 shows current as a function of individual parameters (ϵ,μ,λ). The plots indicate that individual parameters cannot separately bring about current reversals. However, the possibility of current reversals arises due to the combined effect of the three asymmetry parameters considered. We have also observed more number of current reversals than shown in Fig. 8 by fine tuning the parameters.

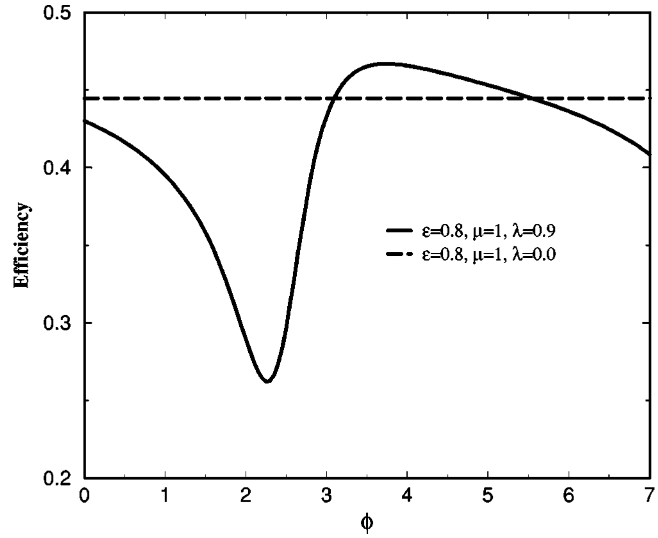

FIG. 7. Efficiency vs ϕ for $\epsilon=0.8$, $\mu=1$, and $L=0.44$ for (i) $\lambda=0.0$ and (ii) $\lambda=0.9$ with fixed $F_0=0.1$ and $T=0.1$

IV. CONCLUSIONS

Using the method of stochastic energetics, we have studied in detail the nature of efficiency, currents, and input energy for temporally asymmetric rocked ratchets. We have considered different cases wherein potential is spatially symmetric or asymmetric and there is frictional inhomogeniety in the medium. We find large efficiency for these rocking ratchet systems, the origin of which can be traced to the suppression of backward motion. The observed efficiency is much higher than the earlier reported values even though the ratchet operates in an intrinsically irreversible domain. The temporal asymmetry parameter also helps in increasing the range of load of operation of the ratchet.

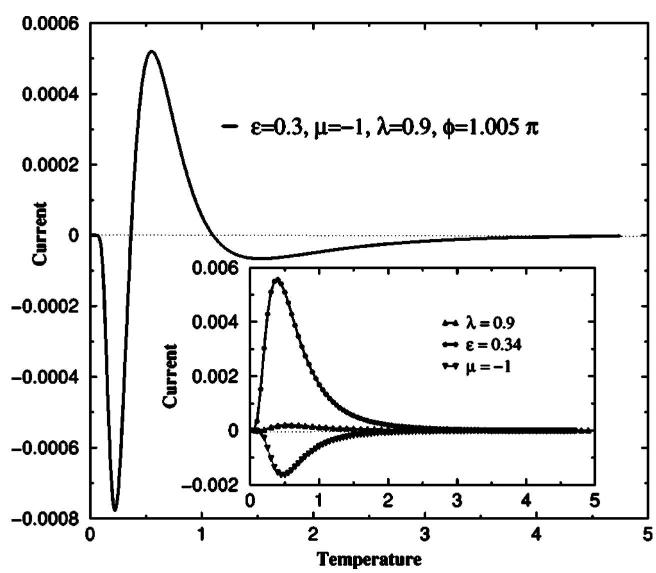

FIG. 8. Current vs temperature curve showing two current reversals for $\mu=-1$, $\lambda=0.9$, and $\epsilon=0.34$ with $\phi=1.005\pi$, $F_0=0.3$, and $L=0$. Inset shows the current in the presence of lone asymmetry parameters (λ,ϵ,μ).

It is worthwhile to explore whether or not, in the high efficiency regime of these ratchet systems, the transport is coherent. Noise-induced currents (transport) are always associated with a dispersion or diffusion. When the diffusion is large, then the quality of transport degrades and the coherence in the unidirectional motion is lost. The coherent transport (optimal transport) refers to the case of large mean velocity at fairly small diffusion and is quantified by a dimensionless Péclet number [23]. This study will be helpful in finding the correlation, if any, between high efficiency and transport coherence. This connection is yet to be explored. We also observe multiple current reversals in the adiabatic limit by the proper fine tuning of different parameters. These reversals are attributed to inherent complex dynamics of the system.

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